

# ON SOLUBILITY OF GROUPS WITH FINITELY MANY CENTRALIZERS

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**ABSTRACT.** For any group  $G$ , let  $\mathcal{C}(G)$  denote the set of centralizers of  $G$ . We say that a group  $G$  has  $n$  *centralizers* ( $G$  is a  $\mathcal{C}_n$ -group) if  $|\mathcal{C}(G)| = n$ . In this note, we prove that every finite  $\mathcal{C}_n$ -group with  $n \leq 21$  is soluble and this estimate is sharp. Moreover, we prove that every finite  $\mathcal{C}_n$ -group with  $|G| < \frac{30n+15}{19}$  is non-nilpotent soluble. This result gives a partial answer to a conjecture raised by A. Ashrafi in 2000.

## 1. Introduction

For any group  $G$ , let  $\mathcal{C}(G)$  denote the set of centralizers of  $G$ . We say that a group  $G$  has  $n$  *centralizers* ( $G \in \mathcal{C}_n$ , or  $G$  is a  $\mathcal{C}_n$ -group) if  $|\mathcal{C}(G)| = n$ . Also we say that  $G$  has a finite number of centralizers, written  $G \in \mathcal{C}$ , if  $G \in \mathcal{C}_n$  for some  $n \in \mathbb{N}$ . Indeed  $\mathcal{C} = \bigcup_{i \geq 1} \mathcal{C}_i$ . It is clear that a group is a  $\mathcal{C}_1$ -group if and only if it is abelian. Belcastro and Sherman in [5], showed that there is no finite  $\mathcal{C}_n$ -group for  $n \in \{2, 3\}$  (while Ashrafi in [2], showed that, for any positive integer  $n \neq 2, 3$ , there exists a finite group  $G$  such that  $|\mathcal{C}(G)| = n$ ). Also they characterized all finite  $\mathcal{C}_n$ -groups for  $n \in \{4, 5\}$ . Tota (see Appendix of [10]) proved that every arbitrary  $\mathcal{C}_4$ -group is soluble. The author in [11] showed that the derived length of a soluble  $\mathcal{C}_n$ -group (not necessarily finite) is  $\leq n$ .

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For more details concerning  $\mathcal{C}_n$ -groups see [1, 2, 3, 4, 11, 12]. In this paper, we obtain a solubility criteria for  $\mathcal{C}_n$ -groups in terms of  $|G|$  and  $n$ .

Our main results are:

**Theorem A.** Let  $G$  be a finite  $\mathcal{C}_n$ -group with  $n \leq 21$ , then  $G$  is soluble. The alternating group of degree 5 has 22 centralizers.

**Theorem B.** If  $G$  is a finite  $\mathcal{C}_n$ -group, then the following hold:

- (1)  $|G| < 2n$ , then  $G$  is a non-nilpotent group.
- (2)  $|G| < \frac{30n+15}{19}$ , then  $G$  is a non-nilpotent soluble group.

Let  $G$  be a finite  $\mathcal{C}_n$ -group. In [5], Belcastro and Sherman raised the question whether or not there exists a finite  $\mathcal{C}_n$ -group  $G$  other than  $Q_8$  and  $D_{2p}$  ( $p$  is a prime) such that  $|G| \leq 2n$ . Ashrafi in [2] showed that there are several counterexamples for this question and then Ashrafi raised the following conjecture (conjecture 2.4): If  $|G| \leq 3n/2$ , then  $G$  is isomorphic to  $S_3, S_3 \times S_3$ , or a dihedral group of order 10. Now by Theorem A, we can obtain that if  $|G| \leq 3n/2$ , then  $G$  is soluble. Therefore Theorem A give a partial answer to the conjecture put forward by Ashrafi.

## 2. Proofs

Let  $n > 0$  be an integer and  $\mathcal{X}$  be a class of groups. We say that a group  $G$  satisfies the condition  $(\mathcal{X}, n)$  ( $G$  is a  $(\mathcal{X}, n)$ -group) whenever in every subset with  $n + 1$  elements of  $G$  there exist distinct elements  $x, y$  such that  $\langle x, y \rangle$  is in  $\mathcal{X}$ . Let  $\mathcal{N}$  and  $\mathcal{A}$  be the classes of nilpotent groups and abelian groups, respectively. Indeed, in a group satisfying the condition  $(\mathcal{A}, n)$ , the largest set of non-commuting elements (or the largest set of elements in which no two generate an abelian subgroup) has size at most  $n$ .

Here we give an interesting relation between groups that have  $n$  centralizers and groups that satisfy the condition  $(\mathcal{A}, n - 1)$ .

**Proposition 2.1.** *Let  $n$  be a positive integer and  $G$  be a  $\mathcal{C}_n$ -group (not necessarily finite). Then  $G$  satisfies the condition  $(\mathcal{A}, n - 1)$ .*

*Proof.* Suppose, for a contradiction, that  $G$  does not satisfy the condition  $(\mathcal{A}, n - 1)$ . Therefore there exists a subset  $X = \{a_1, a_2, \dots, a_n\}$  of  $G$

such that  $\langle a_i, a_j \rangle$  is not abelian, for every  $1 \leq i \neq j \leq n$ . This follows that  $C_G(a_i) \neq C_G(a_j)$  for every  $1 \leq i \neq j \leq n$ . Now since  $C_G(e) = G$ , where  $e$  is the trivial element of  $G$ , we get  $n = |\mathcal{C}(G)| \geq n + 1$ , which is impossible.  $\square$

Note that by easy computation we can see that the symmetric group of degree 4,  $S_4$ , satisfies the condition  $(\mathcal{A}, 10)$ , but  $S_4$  is not a  $\mathcal{C}_{11}$ -group (in fact,  $S_4$  is a  $\mathcal{C}_{14}$ -group). That is, the converse of the above Proposition is not true.

We can now deduce Theorem A.

**Proof of Theorem A.** Clearly every group satisfies the condition  $(\mathcal{A}, n)$  also satisfies the condition  $(\mathcal{N}, n)$ . Thus, by Proposition 2.1,  $G$  satisfies the condition  $(\mathcal{N}, n)$  for some  $n \leq 20$ . Now this statement follows from the main result of [6]. By easy computation we can obtain that the alternating group of degree 5, has 22 centralizers.

Note that Ashrafi and Taeri in [4], proved that, if  $G$  is a finite simple group and  $|\mathcal{C}(G)| = 22$ , then  $G \cong A_5$ . Then they, by this result, claimed that, if  $G$  is a finite group and  $|\mathcal{C}(G)| \leq 21$ , then  $G$  is soluble. Therefore, in view of Theorem A, we gave positive answer to their claim.

Tota in [10, Theorem 6.2]) showed that a group  $G$  belongs to  $\mathcal{C}$  if and only if it is center-by-finite. Therefore it is natural problem to obtain bounds for  $|G : Z(G)|$  in terms of  $n$ .

**Theorem 2.2.** *There is some constant  $c \in \mathbf{R}_{>0}$  such that for any  $\mathcal{C}_n$ -group  $G$*

$$n \leq |G : Z(G)| \leq c^{n-1}.$$

*Proof.* First, by the main result of [9] and Proposition 2.1 we have  $|G : Z(G)| \leq c^{n-1}$ , for some constant  $c$ . To remain the prove we may assume that  $Z(G) \neq 1$ . Since elements in the same coset modulo  $Z(G)$  have the same centralizer, it follows that  $n \leq |G : Z(G)|$ .  $\square$

For the proof of Theorem B, we need the following lemma.

**Lemma 2.3.** *Let  $G$  be a finite  $\mathcal{C}_n$ -group. Then*

$$n \leq \frac{|G| + |I(G)|}{2},$$

where  $I(G) = \{a \in G \mid a^2 = 1\} = \{a \in G \mid a = a^{-1}\}$ .

*Proof.* Since  $C_G(a) = C_G(a^{-1})$ , we can obtain that

$$n \leq |I(G)| + \left| \frac{G - I(G)}{2} \right| \leq \frac{|G| + |I(G)|}{2},$$

as wanted.  $\square$

**Corollary 2.4.** *Let  $G$  be a finite simple  $\mathcal{C}_n$ -group. Then  $3n/2 < |G|$ .*

*Proof.* It is well known that for every simple group we have  $|I(G)| < |G|/3$ . Now the result follows from Lemma 2.3.  $\square$

Here we show that a semi-simple  $\mathcal{C}_n$ -group has order bounded by a function of  $n$ . (Recall that a group  $G$  is semi-simple if  $G$  has no non-trivial normal abelian subgroups.)

**Proposition 2.5.** *Let  $G$  be a semi-simple  $\mathcal{C}_n$ -group. Then  $G$  is finite and  $|G| \leq (n-1)!$ .*

*Proof.* The group  $G$  acts on the set  $A := \{C_G(x) \mid x \in G \setminus Z(G)\}$  by conjugation. By assumption  $|A| = n-1$ . Put  $B = \bigcap_{x \in G} N_G(C_G(x))$ . The subgroup  $B$  is the kernel of this action and so

$$G/B \hookrightarrow S_{n-1}. \quad (*)$$

By definition of  $B$ , the centralizer  $C_G(a)$  is normal in  $B$  for any element  $a \in G$ . Therefore  $a^{-1}a^b \in C_G(a)$  for any two elements  $a, b \in B$ . So  $B$  is a 2-Engel group (see [7]). Now it is well known that  $B$  is a nilpotent group of class at most 3. Now as  $G$  is a semi-simple group, we can obtain that  $B = 1$ . It follows from (\*) that  $G$  is a finite group and  $|G| \leq (n-1)!$ , as wanted.  $\square$

We need the following result, for the proof of Theorem B.

**Theorem 2.6.** (Potter, 1988) *Suppose  $G$  admits an automorphism which inverts more than  $4|G|/15$  elements. Then  $G$  is soluble.*

**Proof of Theorem B.** (1). Suppose, for a contradiction, that  $G$  is nilpotent group, so in particular  $Z(G) \neq 1$ . Now it follows from Theorem 2.2 that  $2n \leq |G|$ , which is contradiction.

(2). From part (1) we obtain that  $G$  is not nilpotent. Since  $|G| < \frac{30n+15}{19}$  and so  $2n > \frac{19|G|-15}{15}$ , Lemma 2.3 implies that

$$|I(G)| \geq 2n - |G| > \frac{4|G|}{15} - 1.$$

On the other hand, since  $I(G)$  is the set of all elements of  $G$  that inverted by the identity automorphism, Theorem 2.6 completes the proof.

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